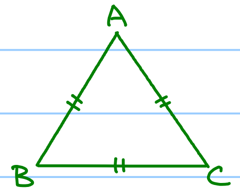
§ 1 Euclid's Elements

1.1 Axiomatic Approach

A "problematic" student.

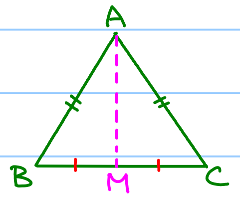
Q (student): Given an equilateral triangle ΔABC, why $\angle A = \angle B = \angle C = 60°$?

A (teacher) $\angle$ sum of Δ + base $\angle$s, isos Δ.



Q: Why base $\angle$s, isos Δ. ?

A. Let M be the mid-pt of BC, and prove $\Delta AMB \cong \Delta AMC$ (SSS)



Q: Why SSS ? (Even worse.

Does a mid-pt of a line segment exist? Why it is unique?

Why are we able to join two distinct points with a line segment? Why it is unique?

. .)

A: . . .

It suggests we should stop at some points!

Axiom: something we accept to be true without further questioning.

Two comments:

1) We should not assume too less / much!

Too less: not much we can deduce.

Too much: some may redundant, may lead contradiction.

(Teacher: Why is "..." true ?

Student: It is assumed to be true.

Teacher: ...)

2) Different sets of axioms may lead to different consequences (different geometries)

Euclid's Elements :

regarded as the prime example of axiomatic method,

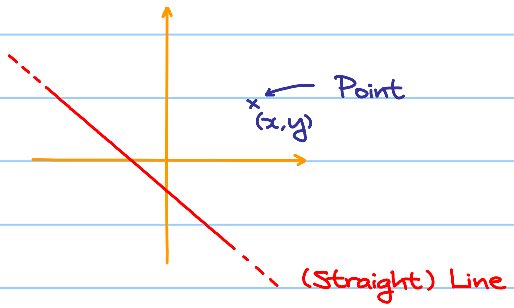
i.e. starting from a small number of self-evident truths (postulates and common notions)

and deducing succeeding results by purely logical reasoning

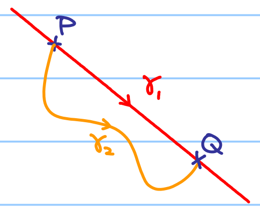
1.2 Different Geometries

1) Plane Geometry

Space = plane $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$



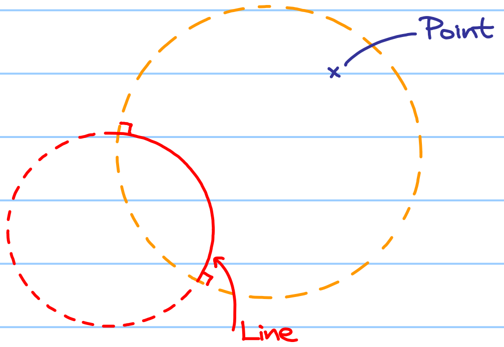
"Straight" in the sense that

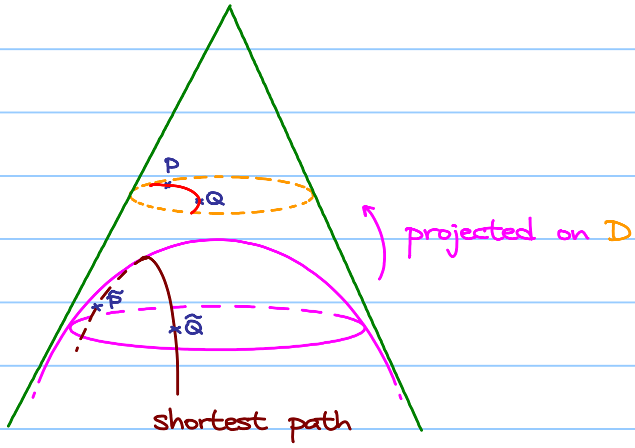


dist $\gamma_1 \leq$ dist γ_2

2) Poincare Disk Model

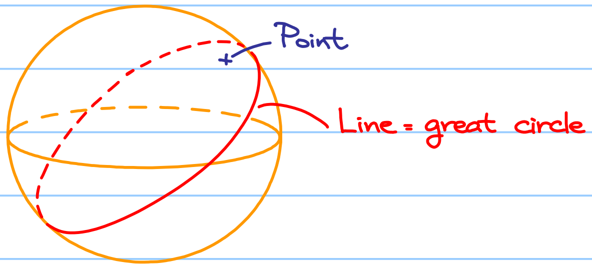
Space = open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$





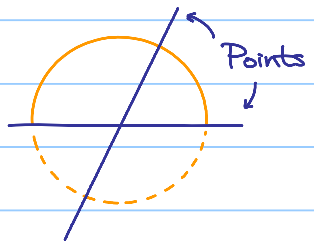
3) Sphereical Geometry

Space = sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$



4) (Real) Projective Space

Space = $\mathbb{RP}^n$ = set of all lines in $\mathbb{R}^{n+1}$ that passes through the origin



Line = collection of points (lines in usual sense)

$\mathbb{RP}^1$ = circle $\approx S^1$

What is $\mathbb{RP}^2$?

Any more "wild" geometry??

Conclusion:

What is a point? What is a line?

It is hard to say!

1.3 Book I of Euclid's Elements

Goal: 1) Appreciate Euclid's axiomatic approach of studying geometry.

2) Find out the parts which are not rigorous from modern point of view.

Content of Book I (see [3])

Definitions 1-23

Postulates 1-5

Common Notions 1-5

Propositions 1-48

1.3 Discussion

Before discussion : (Pretend) Forgetting everything learned before !

If Euclid's (or any other) theory is complete, all propositions can be proved without ambiguity, relying on intuition or diagrams Everytime we think a statement is obvious, it is because we are familarize with plane geometry and we simply test the statement with our inherent knowledge. Just think like Euclid, unless imposing postulate 3, we are not able to construct a circle in the space with a given center and radius

Definitions

Definition · Give precise meaning to the term being defined.

1) Intuition involved in definitions

For example,

Definition 1.10 (definition of right angle) :

> When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right,* and the straight line standing on the other is called a *perpendicular* to that on which it stands.

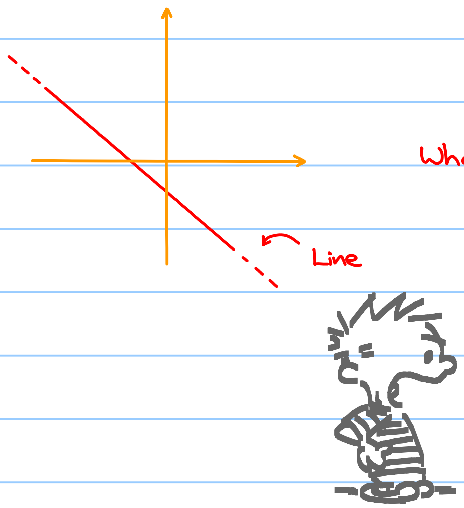
The above defintion is clear enough if we know the meaning of a straight line, an angle and equality of angles

However,

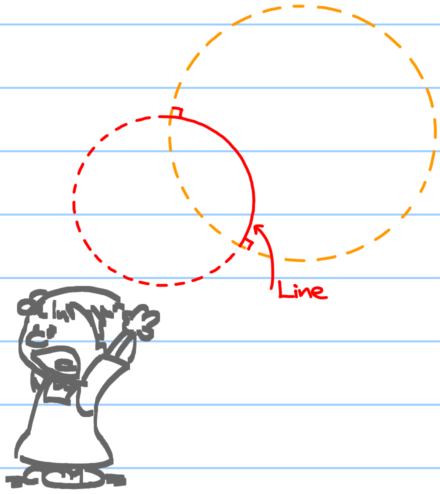
Definition 1.1 (definition of point) : A *point* is that which has no part.

Definition 1.2 (definition of line) : A *line* is breadthless length.

They give no better understanding of point and line.



What is a line ?



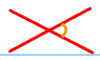
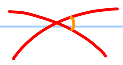
Modern approach : Leave these notions to be undefined.

Regard the space as a set S, elements of S are "points".

Certain subset of S (collection of points) are regarded as "lines".

and if we impose sufficient conditions (axioms), then plane geometry would be the unique model

2) Terminologies

Euclid's Elements			Modern usage
straight-line			line
finite straight-line			line segment
line			curve
rectilinear angle		(made by two staight lines)	angle
angle		(made by two curves)	

3) Equality

- Euclid did not define "equality"
- Refers to congruence of geometrical figures, but also refers to equal in areas.
- Magnitude of the same kind can be:

    compared : equal, less than, greater than

    added or subtracted

(Stop! Think: What is the meaning of the sum of two line segments?)

(suggested by the common notions)

For example, how to define "congruence of line segments"?

Intuitively, two line segments are congruent if they have the same length

However, there is no concept of length of line segment, even concept of real numbers in Euclid's Element.

Equivalence relation will be introduced later to tackle the problem.

Postulates and Common Notions

Postulates and Common Notions : facts that are taken for granted and used as the starting point for logical deduction of theorems

1) Postulates vs Common Notions

Postulates : About geometrical content

Common Notions : About universal nature

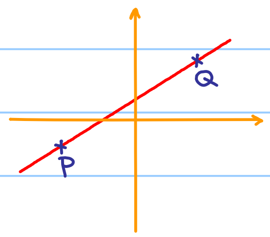
2) Existence and Uniqueness

Postulate 1.1 To draw a straight line from any point to any point.

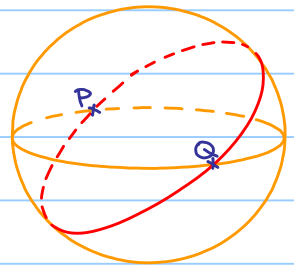
Postulate 1.2 To produce a finite straight line continuously in a straight line.

Postulate 1.3 To describe a circle with any center and radius.

Euclid makes no explicit statement about uniqueness (but actually he used it).



Postulate 1.1



If P and Q are antinodal points, there are infinitely many lines that passes through P and Q.

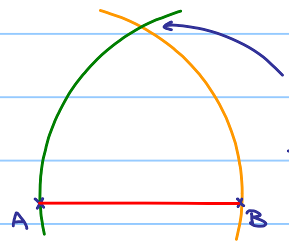
Intersections of Circles and Lines

1) Existence of intersection of two circles

While postulate 1.5 guarantees that two lines will meet under certain conditions, Euclid never tells when two circles will meet.

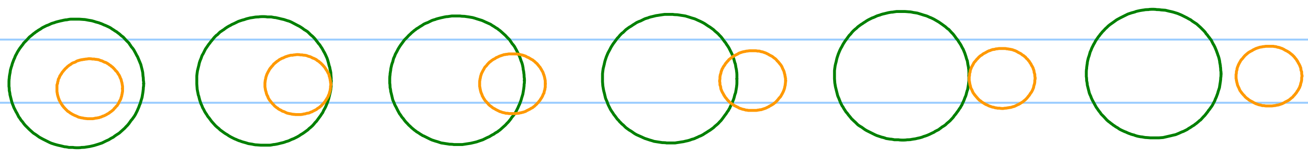
For example,

Proposition 1.1 To construct an equilateral triangle on a given finite straight line.



How to guarantee that there is an interection point?

2) Relative Position



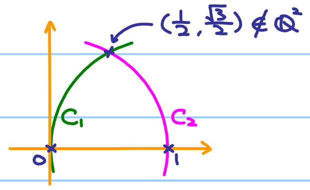
Even for two circles on plane, whether they have intersection, it depends on their relative position.

3) Space

Note that the concept of real numbers was only made until 19th century,

how do we know a plane $= \mathbb{R}^2$? (Why not $\mathbb{Q}^2$?)

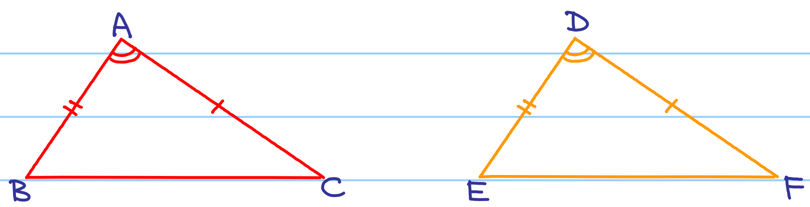
If a plane $= \mathbb{Q}^2$, then C_1 and C_2 have no intersection.



The Method of Superposition

1) Existence of Method of Superposition

Proposition 1.4 If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
Given ΔABC and ΔDEF. If $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$, then $AC = EF$, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$ (known as "SAS").



proof by Euclid.

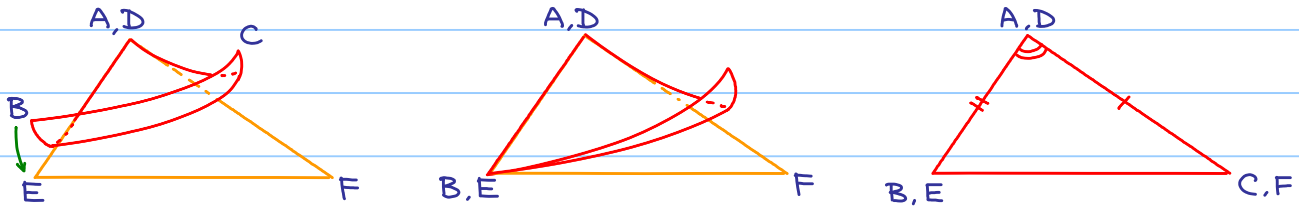
If ΔABC is superposed on ΔDEF, and if the point A is placed on the point D and the straight line AB on DE, then the point B also coincides with E, because AB equals DE.

Again, AB coinciding with DE, the straight line AC also coincides with DF, because $\angle BAC$ equals $\angle EDF$. Hence the point C also coincides with the point F, because AC again equals DF.

But B also coincides with E, hence the base BC coincides with the base EF and equals it. (C.N.4)

Thus the whole ΔABC coincides with the whole ΔDEF and equals it. (C.N.4)

And the remaining angles also coincide with the remaining angles and equal them, $\angle ABC$ equals $\angle DEF$, and $\angle ACB$ equals $\angle DFE$.



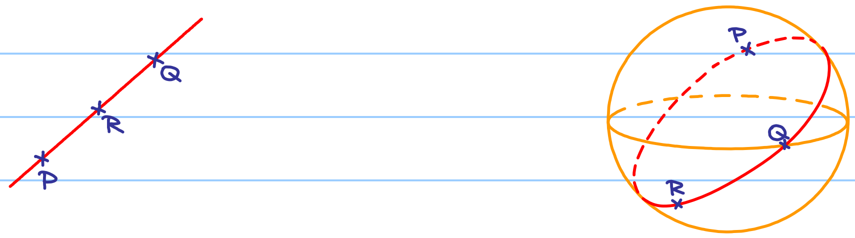
- Euclid does not impose any postulate to allow superposition.
- Euclid is relucant to use superposition (only appears in proposition 1.4 and 1.8).
- Postulate 1.4 That all right angles equal one another.

  It would be unnecessary if superposition is accepted.
- To be precise, superposition is based on rigid motion

Betweenness

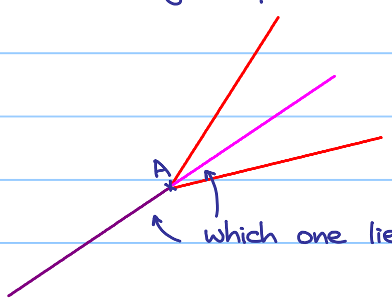
1) What is betweeness?

Euclid makes no statements on explanation of betweenness, such as "one point is between two other on a line",



Which point lies between the other two?

"a line through a point lies inside an angle at that point"



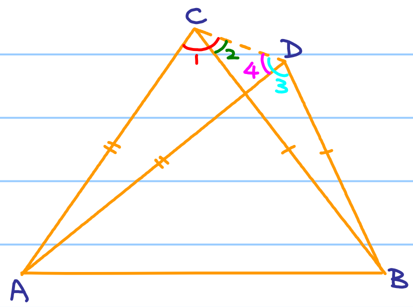
2) Why betweenness it important ?

Some statements may depend on the relative position of points and lines

For example,

Proposition 1.7 Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.
Given a line segment AB. If C and D are points on the same side of AB such that $AC = AD$ and $BC = BD$, then C and D must be the same point.

proof by Euclid.



Suppose $C \neq D$

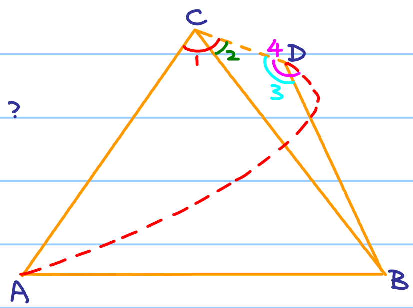
$\angle 2 = \angle 3$ and $\angle 1 = \angle 4$ (prop. 1.5)

$\angle 4 < \angle 3$ (C.N. 5)

$\therefore \angle 1 = \angle 4 < \angle 3 = \angle 2$

which contradicts to that $\angle 2$ is a part of $\angle 1$ (i.e $\angle 1 > \angle 2$ by C.N. 5)

But the proof depends on the diagram!



When we join AD, how do we know AD lie between $\angle 3$?

If AD goes like the red dotted line, then $\angle 3 < \angle 4$ and no contradiction occurs.

The Theory of Parallel

1) Necessity of Fifth Postulate

Postulate 1.5 That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

It looks like a proposition rather than a postulate. Actually, Euclid postponed using it until proving proposition 1.29 (after discussion on triangles and congruence)

Can postulate 1.5 be deduced from the other four postulates ?

If yes, then postulate 1.5 is redundant.

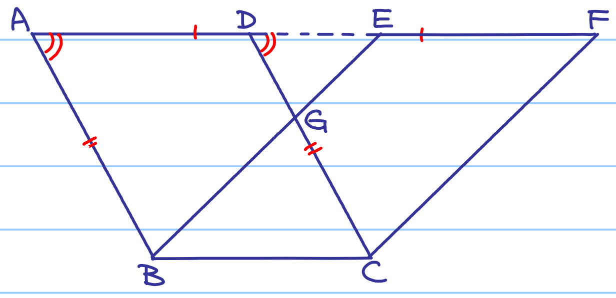
If no, how to prove ? If postulate 1.5 is necessary, that means other geometries may come in when it is removed. (i.e Giving a counter-example of geometry that satisfies postulate 1.1 - 1.4 but not 1.5.)

The Theory of Area

1) What is (How to define) "area"?

Proposition 1.35 Parallelograms which are on the same base and in the same parallels equal one

proof by Euclid:



$AD = BC$ and $EF = BC$ (prop. 1.34)

$\therefore AD = EF$ (C.N. 1)

$AD + DE = EF + DE$ (C.N. 2)

$AE = DF$

$AB = DC$ (prop. 1.34)

$\angle EAB = \angle FDC$ (prop. 1.29 / Corr. $\angle$s, $AB /\!/ DC$)

$\therefore \triangle EAB \cong \triangle FDC$ (prop 1.4 / SAS)

($\therefore$ Area of $\triangle EAB$ = Area of $\triangle FDC$)

Area of $\triangle EAB - \triangle EDG + \triangle GBC$ = Area of $\triangle FDC - \triangle EDG + \triangle GBC$ (C.N. 2 + C.N. 3)

Area of ABCD = Area of EBCF

- Euclid does not have the formula "Area of //gram = Base × Height".
  Even he does not define "length" and "real numbers".
  (That's why he has to construct such a proof!)
- "Area" is not defined by Euclid. However, according to Euclid, area is a quantity that can be added, subtracted and etc.